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Short Communication

Vibration of a circular plate with an attached core

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1. Introduction

The natural frequencies of circular plates are basic in vibration design. Previous literature has been reviewed in Refs. [1–3]. The present paper considers the vibration of a circular plate with a finite, axisymmetric, rigid core attached at the center. If the core has infinite mass, the problem becomes the vibration of an annulus with a clamped interior edge. However, for finite mass, the core affects the vibrational frequency through its translational and rotational inertia.

The interaction of a finite core with a supporting membrane was recently delineated by Wang [4]. It was found that the finite core introduces a slow wobbling mode which is significant for small-core radii. The purpose of this paper is to investigate whether this interesting phenomena would extend to the more difficult plate problem.

2. Formulation

Fig. 1(a) shows the cross section in the equilibrium state. The plate has radius R and is firmly attached to a rigid body of radius bR. The outer edge of the plate may be clamped, simply supported, free or sliding. The classical plate equation is

$$D\nabla^4 w' + \rho \frac{\partial^2 w'}{\partial t^2} = 0, \tag{1}$$

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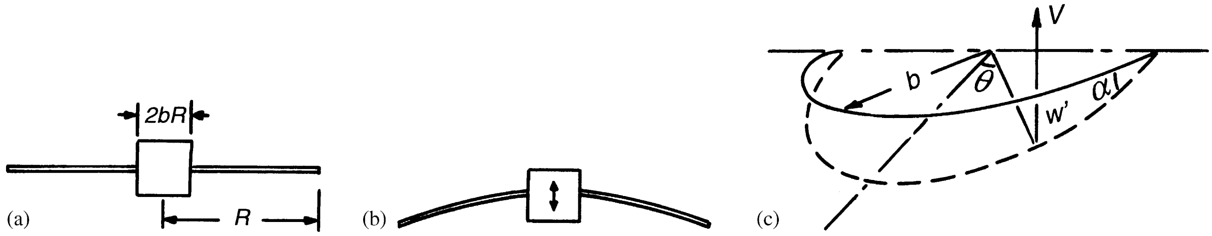

Fig. 1. (a) Cross section of circular plate with a core. (b) Up–down traslational vibration. (c) Anti-symmetric vibration about a diameter. Half the mid plane of the core is shown tilted at an angle α.

where w' is the vertical displacement, D is the flexural rigidity, ρ is the density of the plate. Eq. (1) has the solution

$$w' = Ru(r)\cos(n\theta)\mathrm{e}^{\mathrm{i}\Omega t}, \tag{2}$$

where r is the radius normalized by R, n is the number of nodal diameters and Ω is the frequency. The function u is a linear combination of the Bessel functions $J_n(kr)$, $Y_n(kr)$, $I_n(kr)$, $K_n(kr)$, where

$$k = R\left(\frac{\rho\Omega^2}{D}\right)^{1/4} \tag{3}$$

represents the square root of the normalized frequency. The radial Kirkhoff shear stress is

$$V' = -\frac{D}{R^3}\left[\frac{\partial}{\partial r}\nabla^2 w' + \frac{(1-v)}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w'}{\partial\theta^2}\right)\right], \tag{4}$$

where v is the Poisson ratio. The radial moment is

$$M' = -\frac{D}{R^2}\left[\frac{\partial^2 w'}{\partial r^2} + v\left(\frac{1}{r}\frac{\partial w'}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w'}{\partial\theta^2}\right)\right]. \tag{5}$$

The axisymmetric finite solid core may vibrate in two modes: an up–down translation and a wobbly rotation about a diameter. Fig. 1(b) shows the translational mode for which $n = 0$. It is seen that the slope of the plate is zero at the boundary $r = b$. Thus

$$\frac{\mathrm{d}u}{\mathrm{d}r}(b) = 0. \tag{6}$$

The dynamical equation on the core is

$$2\pi bRV' = m\frac{\partial^2 w'}{\partial t^2}, \tag{7}$$

where m is the mass of the core and the shear and displacement are evaluated at the boundary. Eq. (7) is reduced to

$$2\left[b\frac{\mathrm{d}^3u}{\mathrm{d}r^3}(b) + \frac{\mathrm{d}^2u}{\mathrm{d}r^2}(b) - \frac{1}{b}\frac{\mathrm{d}u}{\mathrm{d}r}(b)\right] - \sigma k^4 u(b) = 0, \tag{8}$$

where

$$\sigma \equiv \frac{m}{\pi \rho R^2} \tag{9}$$

is a mass ratio. The conditions for the rotational mode are more difficult. Fig. 1(c) shows the symmetry plane of the core tilted at an angle α about $\theta = \pi/2$. Continuity in slope gives

$$\frac{\mathrm{d}u}{\mathrm{d}r}(b) = \frac{u(b)}{b}. \tag{10}$$

Balancing the total torque about the rotational axis yields

$$2\int_{-\pi/2}^{\pi/2} V' b^2 R^2 \cos\theta \, \mathrm{d}\theta - 2\int_{-\pi/2}^{\pi/2} M' b R \cos\theta \, \mathrm{d}\theta = I \frac{\mathrm{d}^2 \alpha}{\mathrm{d}t^2}, \tag{11}$$

where I is the rotational moment of inertia of the core. After some work, Eq. (11) is simplified to

$$b^3 \frac{\mathrm{d}^3 u}{\mathrm{d}r^3}(b) - 3b \frac{\mathrm{d}u}{\mathrm{d}r}(b) + 3u(b) = \eta k^4 u(b), \tag{12}$$

where

$$\eta \equiv \frac{I}{\pi \rho R^4} \tag{13}$$

is a moment of inertia ratio. Since a solid body cannot have higher modes, the boundary conditions for $n > 1$ are

$$u(b) = 0, \quad \frac{\mathrm{d}u}{\mathrm{d}r}(b) = 0. \tag{14}$$

Together with the boundary conditions at the edge of the plate, the natural frequencies are then determined.

3. The clamped plate

The general solution is

$$u(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr). \tag{15}$$

The boundary conditions at the plate edge are

$$u(1) = 0, \tag{16}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r}(1) = 0. \tag{17}$$

For the translational mode Eqs. (6), (8), (16) and (17) are applied to Eq. (15). The determinant of the coefficients C_n is set to zero, yielding a transcendental equation in the frequency k. The roots are found by a simple bisection algorithm. Fig. 2(a) shows the results for the axisymmetric or $n = 0$ mode. It is seen that frequency rises with core radius b in bands. There are voids between the bands where certain frequencies are unrealizable. When $\sigma = \infty$ the effect of the core is equivalent

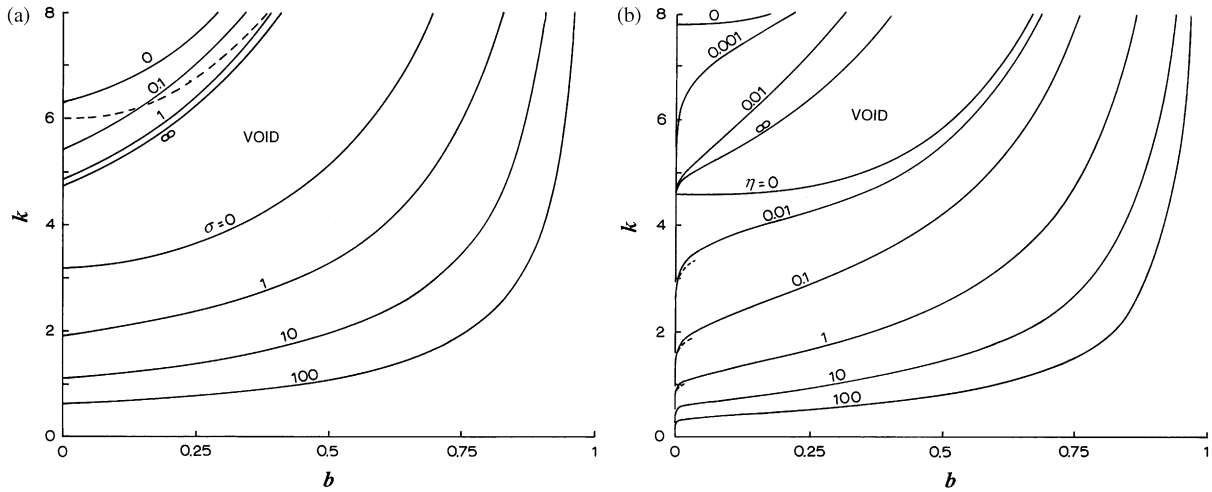

Fig. 2. Frequencies of plate with clamped boundary: (a) $n=0$ mode, (b) $n=1$ mode. The dashed curve is the $n=2$ mode. Dotted curves are from the asymptotic solution Eq. (19).

to a clamped inner boundary. Our results compare well with the results of the clamped–clamped annulus (e.g. Ref. [5]). When $\sigma = 0$ the inner boundary becomes a sliding or movable boundary. Of interest is the behavior when the core shrinks to a point mass. The characteristic equation is

$$8[I_1(k)J_0(k) + I_0(k)J_1(k)] - \sigma k^2\{\pi[I_1(k)Y_0(k) + I_0(k)Y_1(k)] + 2[J_1(k)K_0(k) - J_0(k)K_1(k)] + 4/k\} = 0. \tag{18}$$

Thus, for $\sigma = 0$, the frequencies start at 3.196, 6.306, etc. and for $\sigma = \infty$ the frequencies start at 0, 4.768, 7.871, etc. We mention that the axisymmetric case was considered by Handelman and Cohen [6], but their computed results seem to be erroneous. The frequencies for the asymmetric $n = 1$ mode are shown in Fig. 2(b). Notice that for a finite core the lowest frequency rises singularly from zero when $b = 0$. Using Bessel function expansions for small arguments, we find asymptotically

$$k \sim \left(\frac{4}{\eta|\ln b|}\right)^{1/4}. \tag{19}$$

Therefore, for small core radii, the fundamental frequency may be dominated by a slow wobbly $n = 1$ mode. Again, using expansions, we find for non-zero η that the higher frequencies all rise singularly from discrete points governed by the roots of

$$J_1(k)[I_0(k) + I_2(k)] - I_1(k)[J_0(k) - J_2(k)] = 0. \tag{20}$$

i.e. 4.525, 7.734, etc.

4. The simply supported plate

The boundary conditions at the plate edge are zero displacement Eq. (16), and zero moment

$$\frac{d^2u}{dr^2}(1) + v\left[\frac{du}{dr}(1) - n^2u(1)\right] = 0. \tag{21}$$

In this case the Poisson ratio is a factor. The Poisson ratio ranges from 0.2 for concrete, to 0.3 for metals and 0.4 for polymers. The frequencies for the axisymmetric $n = 0$ mode are shown in Fig. 3(a). For $b = 0$ the frequencies are governed by

$$2v[I_1(k)J_0(k) + I_0(k)J_1(k)] + k[2I_0(k)J_0(k) + I_2(k)J_0(k) - I_0(k)J_2(k)] = 0. \tag{22}$$

Although v is present in Eq. (22), its effect on the frequency is not large. For example, the first frequency for $\sigma = 0$ starts at 2.187, 2.222, and 2.253 for v =0.2, 0.3, 0.4, respectively. The first non-zero frequency for $\sigma = \infty$ starts at 3.831, 3.866, and 3.848 for v =0.2, 0.3, 0.4, respectively. Fig. 3(b) shows the $n = 1$ mode, $v = 0.3$. The asymptotic form for small b is the same as Eq. (19). The higher modes start from the roots of

$$2v\{J_1(k)[I_0(k) + I_2(k)] - I_1(k)[J_0(k) - J_2(k)]\} + k[6I_1(k)J_1(k) + I_3(k)J_1(k) - I_1(k)J_3(k)] = 0, \tag{23}$$

i.e. 3.729, 6.963, etc for $v = 0.3$.

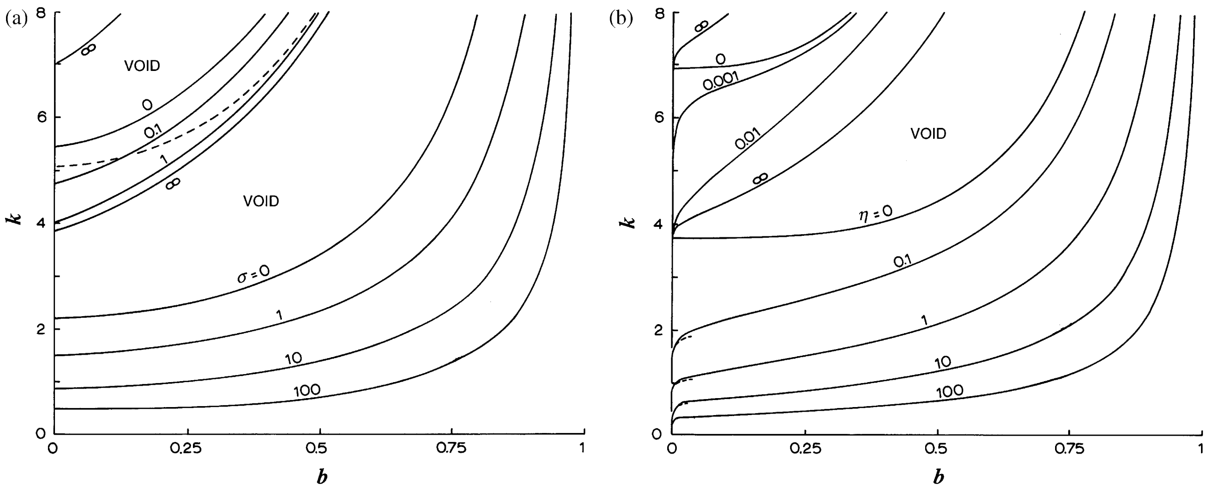

Fig. 3. Frequencies of plate with simply supported boundary: (a) $n=0$ mode, (b) $n=1$ mode. The dashed curve is the $n=2$ mode. Dotted curves are from the asymptotic solution Eq. (19).

5. The plate with free edge

The edge boundary conditions are zero moment, Eq. (21), and zero shear:

$$\frac{d^3u}{dr^3}(1) + \frac{d^2u}{dr^2}(1) - [1 + n^2(2 - \nu)]\frac{du}{dr}(1) + n^2(3 - \nu)u(1) = 0. \tag{24}$$

The results for the axisymmetric modes are shown in Fig. 4(a). There are some major differences when the outer edge is free. Firstly, unlike Figs. 2(a) and 3(a), a large void exists for low frequency. For $\sigma = \infty$ the frequency starts from $k = 1.920$, 1.937, 1.953 for $\nu = 0.2$, 0.3, 0.4, respectively. The $\sigma = 0$ curve starts from 2.962, 3.001, 3.036 for the same sequence of ν values. Secondly, the $n = 2$ mode, independent of σ and shown by the dashed lines, is significant at low frequencies (the starting frequencies are 2.378, 2.315 and 2.241, respectively). Note that the fundamental mode is the $n = 2$ mode for the full plate, while here the fundamental mode depends on the mass ratio σ.

The antisymmetric $n = 1$ mode is shown in Fig. 4(b). Again there is a void region for low k. The singular rise for the first band is found asymptotically to be independent of ν:

$$k \sim \frac{2}{|\ln b|^{1/4}}\left(1 + \frac{1}{4\eta}\right)^{1/4}. \tag{25}$$

Southwell [7] studied the clamped-free plate in the limit of small b, and found a formula equivalent to Eq. (25) with $\eta \to \infty$. The curves for the second band also start singularly, from 4.510, 4.525, 4.539 for $\nu = 0.2$, 0.3, 0.4, respectively.

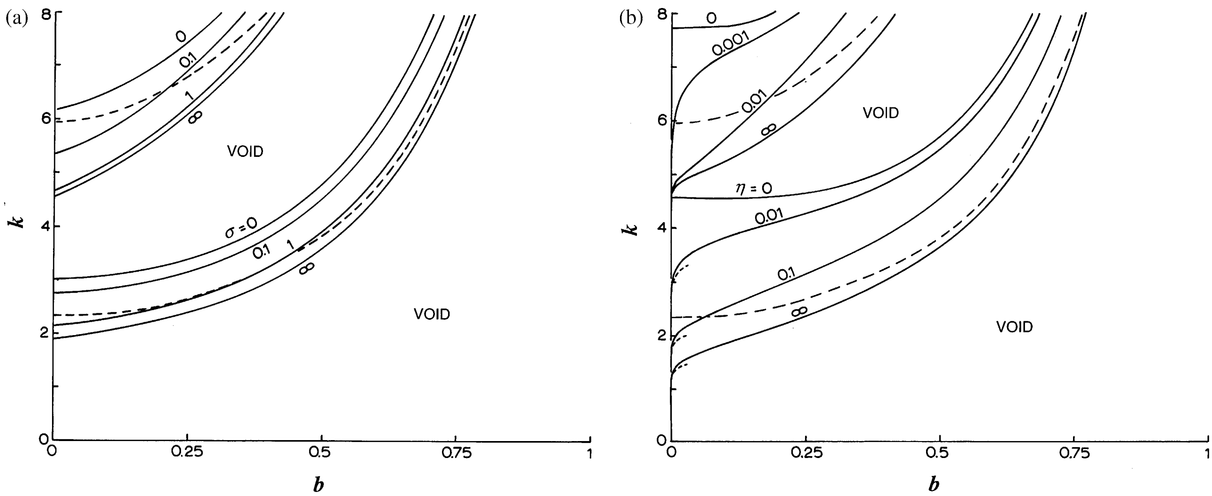

Fig. 4. Frequencies of plate with free boundary: (a) $n = 0$ mode, (b) $n = 1$ mode. The dashed curves are the $n = 2$ mode. Dashed curves are the $n = 2$ mode. Dotted curves are from the asymptotic solution Eq. (25).

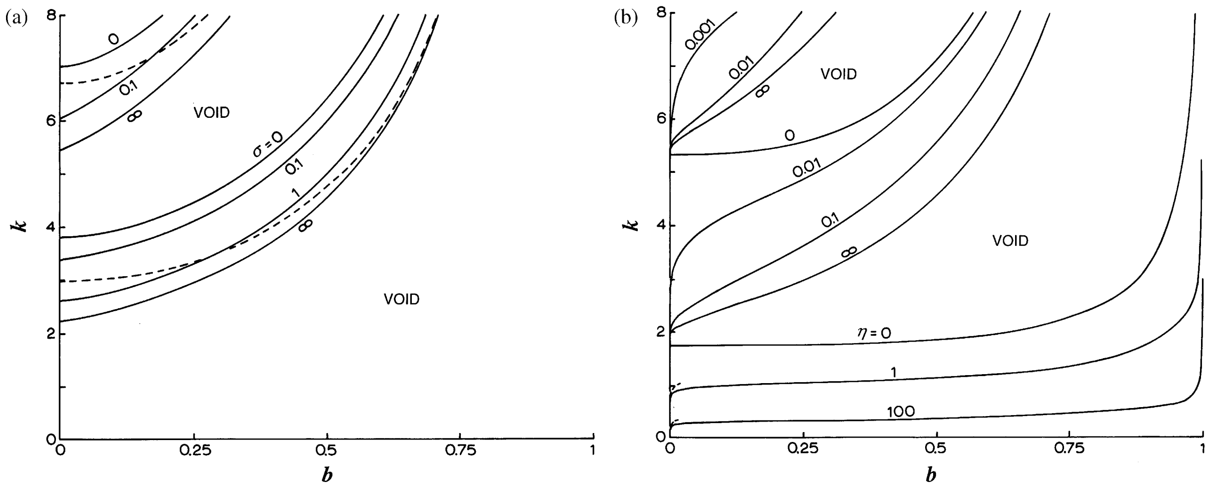

Fig. 5. Frequencies of plate with sliding boundary: (a) $n=0$ mode, (b) $n=1$ mode. The dashed curves are the $n=2$ mode. Dotted curves are from the asymptotic solution Eq. (19).

6. The plate with a sliding edge

The sliding edge or movable edge is important for moving parts such as piston heads. The boundary conditions are zero slope, Eq. (17), and zero shear, Eq. (24). Fig. 5(a) shows the $n=0$ mode which is independent of Poisson's ratio ν. The lowest $\sigma=\infty$ curve starts from 2.233, while the $\sigma=0$ curve starts from 3.832. As in the free-edge case, the $n=2$ mode is important in comparison, starting from 2.950, 2.964, 2.977 for $\nu=0.2$, 0.3, 0.4, respectively. Fig. 5(b) shows the $n=1$ mode. The frequencies of the first band are much lower than the $n=0$ or $n=2$ modes that they would be fundamental modes for all b. For small b the asymptotic formula is given by Eq. (19). The second band starts from $k=1.742$, 1.756, 1.769 for the sequence of ν's. The third band starts from $k=5.329$ almost independent of ν.

7. Conclusions

In this paper, we studied a plate with an attached rigid mass for the first time. The problem differs from that of the stepped plate since the rigid mass is not thin and has substantial rotary inertia. Similar to the vibration of a membrane with a rigid core [4], the vibration of a plate with a rigid core has a slow, antisymmetric wobbly mode and void regions in the frequency-core size graphs. There are also differences. The plate vibrations are influenced by the edge conditions and the Poisson's ratio. The frequencies of the axisymmetric mode of the membrane rise singularly from zero for small b, while those of the plate are discrete constants when b is zero. On the other hand, the frequencies of the antisymmetric modes of the membrane rise smoothly, while those of the plate rise singularly.

Our extensive frequency graphs should be useful in the design of plates supporting central solid masses.

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